Determination of forward and futures prices

_Foul cankering rust the hidden treasure frets,  
But gold that's put to use more gold begets._

—William Shakespeare, Venus and Adonis, 1593

Overview

- Investment assets vs. consumption assets
- Short selling
Assumptions and notation

Forward price of an investment asset
Known income
Known yield
Valuing forward contracts
Are forward prices and futures prices equal?
Futures prices of stock indices
Forward and futures prices on currencies
Futures on commodities
The cost of carry
Delivery options
Futures prices and expected future spot prices

Summary of formulae

Table 5.1. Summary table of formulae used to find the forward price and the value of a forward contract, for the three cases in which there is no income, a known income with present value \( I \), and a known yield \( y \).

<table>
<thead>
<tr>
<th>Asset</th>
<th>Forward / futures price</th>
<th>Value of long forward contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>No income</td>
<td>( S_0 e^{rT} )</td>
<td>( S_0 - K e^{-rT} )</td>
</tr>
<tr>
<td>Income of present value ( I )</td>
<td>( (S_0 - I) e^{rT} )</td>
<td>( S_0 - I - K e^{-rT} )</td>
</tr>
<tr>
<td>Yield ( q )</td>
<td>( S_0 e^{(r-q)T} )</td>
<td>( S_0 e^{-qT} - K e^{-rT} )</td>
</tr>
</tbody>
</table>

Introduction

- Relate forward, futures prices to spot of underlying
- Forwards easier than futures (Why?)
- Forward \( \approx \) future (When?)
- General results relate fwd to spot
- Specific
  - stock indices
  - FX
  - commodities
  - (IR, next chapter)
**Investment assets vs. consumption assets**

- **Definition 5.1.** An *investment asset* is an asset that is held primarily for investment.
  - E.g. stocks, bonds, gold
  - Not have to be exclusively for investment e.g. silver

- **Definition 5.2.** A *consumption asset* is an asset that is held primarily for consumption.
  - Not held for investment
  - E.g. copper, oil, pork bellies

**Short selling**

- “Shorting”
- Possible for some investment assets
- Procedure
  - Investor instruct broker
  - Broker borrow shares another client
  - Sells in market
  - Wait
  - Investor buys shares and returns
- Investor profits if the share price _____________.
- “Short-squeeze”
  - broker _________________.
  - investor _________________.
- What about income due to client (“stock lender”), e.g. dividends or interest?

**Example**

**Example 5.1.** An investor shorts 500 shares in April, when the price per share is $120 and closes out the position in July, when the price is $100. A dividend of $1 per share is paid in May. What is the net gain? What would be the loss for an investor who took the equivalent long position?

\[
P&L = \begin{cases} 
500 \times (120 - 100) - 1 & \text{Buy for less than sold Reimburse owner} \\
-500 \times (100 - 120) + 1 & \text{Sell for more than bought Receive div} 
\end{cases} = 500 \times 19 = 9500
\]

**Margin account**

- Does an investor with short sale position have to maintain a margin? Yes □ No □
US shares can only be shorted on an **uptick**

### Assumptions and notation

**Assumptions**
- For some market participants:
  - a. No transaction costs
  - b. Same tax
  - c. No bid-ask spread for interest
  - d. Arb opps exploited

**Notation**

- $T$ time until delivery
- $S_0$ price of underlying
- $F_0$ forward or futures price, today
- $r$ risk-free rate of interest
- $I$ present value of income received during the life of a forward contract
- $q$ average yield per annum on asset during life of forward contract (ct's cmpd)

### Forward price of an investment asset

**Forward prices and spot prices example**
- Recall Chapter 1

**Example 5.2.** A stock pays no dividends and costs $60. The rate for risk-free borrowing and investing is 5% per annum. What is the 1-year forward price of the stock?

$60$ grossed up at 5% for 1 year or $60 \times 1.05 = 63$

*Why? If forward price*   
- *More, say $67, borrow $60, buy one share, sell forward for $67^{\frac{1}{1}\text{year}}$ pay off loan.*

Net profit $4$
• Less, say $58, sell one share, invest $60, buy forward for $58 \上赛季 buy back asset; Net profit $5

Remark
- Take opposite positions in the spot and the forward markets

Forward contract on an investment asset that provides no income

Proposition

Proposition 5.3. The initial forward price $F_0$ and spot price $S_0$ for an investment asset that pays no dividend are related by $F_0 = S_0 e^{rT}$, where...

\[ F_0 = S_0 e^{rT} \]  
(5.1)

- In general $F_t = S_t e^{r(T-t)}$
- Symbols defined in notation box, above

Proof
- The forward price is the ________________ in a _________________ such that ________________.
- $\Rightarrow$ take $K = F_0$

Case $F_0 > S_0 e^{rT}$:

Table 5.2. Arbitrage opportunity strategy when forward price is relatively expensive.

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Holding</th>
<th>Value at 0</th>
<th>Value at T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock</td>
<td>1</td>
<td>$S_0$</td>
<td>$S_T$</td>
</tr>
<tr>
<td>Bank/bond</td>
<td>$-S_0$</td>
<td>$-S_0$</td>
<td>$-S_0 e^{rT}$</td>
</tr>
<tr>
<td>Forward</td>
<td>$-1$</td>
<td>0</td>
<td>$-(S_T - F_0)$</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>0</td>
<td>$F_0 - S_0 e^{rT}$</td>
</tr>
</tbody>
</table>

Case $F_0 < S_0 e^{rT}$:

Table 5.3. Arbitrage opportunity strategy when forward price is relatively cheap.

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Holding</th>
<th>Value at 0</th>
<th>Value at T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock</td>
<td>$-1$</td>
<td>$-S_0$</td>
<td>$-S_T$</td>
</tr>
<tr>
<td>Bank/bond</td>
<td>$S_0$</td>
<td>$S_0$</td>
<td>$S_0 e^{rT}$</td>
</tr>
<tr>
<td>Forward</td>
<td>1</td>
<td>0</td>
<td>$(S_T - F_0)$</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>0</td>
<td>$S_0 e^{rT} - F_0$</td>
</tr>
</tbody>
</table>
Conclusion

- Zero investment leading to certain positive reward in each case is an arbitrage.
- Therefore, because by assumption arbitrage is impossible, equality $F_0 = S_0 e^{rT}$ must hold.

Remarks

- If we assume constant, deterministic interest rates, it does not matter whether we use a bank account or a zero coupon bond as the risk-free instrument in our strategy.
- Examples and exercises in Hull using:
  - stocks tend to assume deterministic, constant $r$ (i.e. a flat yield curve)
  - bonds may require a non-trivial term structure, in which case the risk-free hedging instrument(s) will be (a) zero-coupon bond(s) (e.g. below)

Short sales not possible

- Short sales not possible for all investment assets
- Ability to short asset not essential
- Do require significant number of people holding for investment
- If forward price too low, attractive to adopt
  - {...} position in forward and
  - {...} position in spot,

which causes the forward price to {...} relative to spot.

Known income

Example

Example 5.3. A coupon-bearing bond is worth $900. A long forward contract on the bond expires in 9 months. A coupon payment of $40 is expected after 4 months. The 4-month and 9-month (continuously compounded) interest rates are 3% and 4%, respectively.

- Find strategies to exploit the arbitrage opportunities that exist when the forward price is $870 and $910.
- Find a zero initial cost strategy in the coupon bearing bond, the forward contract on it and zero-coupon bonds of maturities 4 and 9 months so as to establish the forward price. Tabulate the values of the holdings in the different assets at each time.

Present value of coupon income

$I = 40 e^{-0.03 \times \frac{4}{12}} = 39.602$
Forward price is $870
Forward price is cheap ⇒ Buy ........................, sell ........................
........................ $900 from .......... bond;  ............. a forward contract
Invest
$39.602 at 3% pa for 4 months ← Use at 4 mo to ........................
$860.40 at 4% pa for 9 months
Credit at 9 mos is $860.40 e^{0.04\cdot 0.75} = $886.60
However, purchase of bond costs $870 (why?)
Profit to arbitrageur $886.60 – 870 = $16.60

Forward price is $910
Forward price is expensive ⇒ Buy ........................, sell ........................
........................ $900 to .......... bond; and .............. a forward contract
Borrow
$39.602 at 3% pa for 4 months ← Pay off at 4 mo with ........................
$860.40 at 4% pa for 9 months
Amount owing at 9 mos is $860.40 e^{0.04\cdot 0.75} = $886.60
However, sale of bond earns $910 (why?)
Profit to arbitrageur $910 – 886.60 = $23.40

### Table

<table>
<thead>
<tr>
<th>Holding</th>
<th>Value 0</th>
<th>Value 4</th>
<th>Value 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bc</td>
<td>1</td>
<td>S_0</td>
<td>S_0 + I \beta_4</td>
</tr>
<tr>
<td>B_4</td>
<td>-I</td>
<td>-I</td>
<td>-I \beta_4</td>
</tr>
<tr>
<td>B_9</td>
<td>-(S_0 - I)</td>
<td>-(S_0 - I)</td>
<td>- (S_0 - F_0)</td>
</tr>
<tr>
<td>F</td>
<td>-1</td>
<td>0</td>
<td>- (S_0 - F_0)</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>0</td>
<td>F_0 - (S_0 - I) \beta_9</td>
</tr>
</tbody>
</table>

where \( \beta_4 = e^{r_4 \cdot \frac{4}{12}} \), and \( \beta_9 = e^{r_9 \cdot \frac{9}{12}} \)

- Remark: forward contract is not to buy the asset as it is now, but how it will be after the income has been paid
- This effectively a dynamic strategy because the dividends change the bond holding at the 4 month mark

### Proposition

**Proposition 5.4.** The forward price \( F_0 \) and spot price \( S_0 \) for an investment asset that pays a known income are related by \( F_0 = (S_0 - I) e^{rT} \), where...
\[ F_0 = (S_0 - I) e^{rT} \] (5.2)

Proof
- Take price written into forward contract to equal forward price: \( K = F_0 \) (Why?)

Case \( F_0 > (S_0 - h) e^{rT} \)

Table 5.4. Arbitrage opportunity strategy when forward price is relatively expensive.

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Holding</th>
<th>Value at 0</th>
<th>Value at T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock</td>
<td>1</td>
<td>( S_0 )</td>
<td>( S_T )</td>
</tr>
<tr>
<td>Bank/bond</td>
<td>( \left( \frac{1}{1+rT} \right) ) ( S_0 - I ) -</td>
<td>( -(S_0-I)+I )</td>
<td>( -(S_0-I) e^{rT} )</td>
</tr>
<tr>
<td>Forward</td>
<td>-1</td>
<td>0</td>
<td>( -(S_T-F_0) )</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>0</td>
<td>( F_0-(S_0-I) e^{rT} )</td>
</tr>
</tbody>
</table>

- Zero cost strategy gives rise to a certain profit at time \( T \). This is an arbitrage opportunity.

Case \( F_0 < (S_0 - h) e^{rT} \)

Table 5.5. Arbitrage opportunity strategy when forward price is relatively cheap.

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Holding</th>
<th>Value at 0</th>
<th>Value at T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock</td>
<td>-1</td>
<td>-( S_0 )</td>
<td>-( S_T )</td>
</tr>
<tr>
<td>Bank/bond</td>
<td>( \left( \frac{1}{1+rT} \right) ) ( S_0 - I ) +</td>
<td>( (S_0-I)+I )</td>
<td>( (S_0-I) e^{rT} )</td>
</tr>
<tr>
<td>Forward</td>
<td>1</td>
<td>0</td>
<td>( (S_T-F_0) )</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>0</td>
<td>( (S_0-I) e^{rT} - F_0 )</td>
</tr>
</tbody>
</table>

Conclusion
- Zero investment leading to certain positive reward in each case is an arbitrage opportunity.
- Therefore, equality \( F_0 = (S_0 - I) e^{rT} \) must hold

Example

Example 5.4. A stock has value $50. On the stock can be traded a 10-month forward contract. The flat yield curve is at 8% per annum. Dividends of $0.75 per share are expected after 3, 6, and 9 months.
- Find the PV of the divs
- Find the forward price
\[ I = 0.75 (e^{-0.08 \cdot \frac{3}{12}} + e^{-0.08 \cdot \frac{6}{12}} + e^{-0.08 \cdot \frac{9}{12}}) = 2.162 \]

\[ F_0 = (50 - 2.162) e^{0.08 \cdot \frac{9}{12}} = 51.14 \]

**Known yield**

**Proposition 5.5.** The forward price and spot price for an investment asset that pays dividends at a constant rate \( q \) are related by 
\[ F_0 = S_0 e^{(r-q)T} \]

(5.3)

- General relationship for time \( t \) is 
  \[ F_t = S_t e^{(r-q)(T-t)} \]
- Course so far – static, buy and hold strategies
- Proof will require our first dynamic trading strategy: adjust our holdings over time
- This is a deterministic strategy; we know in advance what the holdings will be – unlike delta-hedging for hedging a call options (see other courses), which is dynamic and stochastic

**Proof**
- Take price written into forward contract to equal forward price: 
  \[ K = F_0 \] (Why?)

**Case \( F_0 > (S_0 - I) e^{rT} \)**

**Table 5.6.** Appropriate strategy to derive the forward price for an asset that pays dividends at a constant rate.

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Holding at ( t )</th>
<th>Value at 0</th>
<th>Value at ( t )</th>
<th>Value at ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock</td>
<td>( e^{-q(T-t)} )</td>
<td>( S_0 e^{-qT} )</td>
<td>( S_t e^{(r-q)(T-t)} )</td>
<td>( S_T )</td>
</tr>
<tr>
<td>Bank/bond</td>
<td>( -S_0 e^{-qT} e^{rt} )</td>
<td>( -S_0 e^{-qT} e^{rt} )</td>
<td>( -S_0 e^{-(r-q)T} )</td>
<td>( -(S_T - F_0) )</td>
</tr>
<tr>
<td>Forward</td>
<td>(-1)</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>Total</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( F_0 - S_0 e^{(r-q)T} )</td>
</tr>
</tbody>
</table>

- Holding changes over time
  - Stock – reinvestment of dividends
  - Bank account – usual exponential growth (i.e. usual buy and hold situation)

**Case \( F_0 < (S_0 - I) e^{rT} \)**
- As for Case \( F_0 > (S_0 - I) e^{rT} \), except with all signs reversed.

**Conclusion**
In each case a zero cost investment gives rise to a certain, positive payoff, which is an arbitrage opportunity.

However, by assumption, arbitrage is forbidden. Hence, equality must hold:

\[ F_0 = (S_0 - I) e^{rT} \]

Remarks

- Not in Hull.
- Values at \( t \) are not required for the proof, but help to see the reasoning behind the values at \( T \).

Figure 5.1: Relationship between spot price and forward/futures price as delivery period \( T = 2 \), is approached.

Cases:

- zero or small asset yield (lhs) and
- asset yield exceeds risk free interest rate (rhs).

Example 5.5. An asset of price $25 is expected to pay a dividend stream equal to 2% of the asset price during a 6-month period. The risk-free rate is 10% per annum. Convert the yield to continuous compounding and thereby find the 6-month forward price on the asset.

Yield is 2 × 0.02 = 4% per annum with semiannual compounding:

\[
R = m \ln(1 + \frac{r_m}{m}) = 2 \ln(1 + \frac{0.04}{2}) = 3.96 \%
\]

\[
F_0 = S_0 e^{r(1 - q)T} = $25 e^{(0.04 - 0.0396)0.5} = $25.77
\]
Valuing forward contracts

Notation

\[ f \] value of forward contract today
\[ K \] delivery price

Proposition

Proposition 5.6. The value, \( f \), of a long forward contract is given by

\[ f = (F_0 - K) e^{-rT} \]

where ...

\[ f = (F_0 - K) e^{-rT} \]  \hspace{1cm} (5.4)

- Holds for long positions in investment and consumption assets
- Value of forward contract, when delivery price in contract is the forward price, when it is entered is .......... (Why?)
- Thereafter current forward price is unlikely to match delivery price
- Value of contract, \( f \), typically non-zero, positive or negative
- Proof analogous to FRA argument

Proof

- Compare values of long forwards with delivery prices \( F_0 \) and \( K \) (otherwise identical)
- Consider the following long-short strategy (Why? Hint: which variables are uncertain at time 0?)

<table>
<thead>
<tr>
<th>Forward (delivery price)</th>
<th>Holding</th>
<th>Value at 0</th>
<th>Value at T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward (( K ))</td>
<td>+1</td>
<td>( f )</td>
<td>( S_T - K )</td>
</tr>
<tr>
<td>Forward (( F_0 ))</td>
<td>-1</td>
<td>0</td>
<td>(-(S_T - F_0))</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>( f )</td>
<td>( F_0 - K )</td>
</tr>
</tbody>
</table>

- Certain fee of \( f \) at \( t = 0 \), earns certain reward of \( F_0 - K \) at \( t = T \)
- \( \Rightarrow f = (F_0 - K) e^{-rT} \)

Remarks

- What is the value of a short forward contract with delivery price \( K \)?
Price a forward contract by assuming that the final price of the asset at delivery, \( S_T \), always turns out to be the forward price \( F_0 \).

- Of course, \( S_T \) will never be precisely \( F_0 \). This is just a pricing trick.

**Risk neutral pricing**

**Code**

**Code 2**

**Output**

![Log normal probability density functions for objective and risk-neutral probability measures](image)

- In other courses (FM02) we learn that price of option is its replication cost
- Replication cost is the expectation of the discounted payoff
- Special probability measure
- Not the average replication cost, but the pathwise replication cost, every time!
- Our results for forward and futures prices can be obtained this way too
- E.g. “Black-Scholes” assumptions: geometric Brownian motion etc. (fig. above)
- Our results strong; model independent

**Forward contract values in terms of \( S_0 \)**

**Proposition 5.7.** The value, \( f \), of a long forward contract is given by \( f = S_0 - K e^{-rT} \), where ...

\[
 f = S_0 - K e^{-rT} \tag{5.5}
\]

- Proof: (5.1) in (5.4)
- Similarly

**Proposition 5.8.** The value, \( f \), of a long forward contract on an investment asset that provides a known income is given by \( f = S_0 - I - K e^{-rT} \), where ...
Proposition 5.9. The value, \( f \), of a long forward contract on an investment asset that provides a known yield at rate \( q \) is given by:

\[
f = S_0 e^{-qT} - K e^{-rT}
\]

(5.7)

Proof: (5.3) in (5.4)

Are forward prices and futures prices equal?

**Discussion**

- No, in general
- Yes
  - Risk-free rate deterministic (possibly non-flat yield curve)
  - Special case: constant, with flat yield curve – we prove
- Real world, IRs stochastic

**Argument**

- Consider \( \rho := \text{Corr}(S, r) > 0 \)
- \( S \uparrow \Rightarrow r \uparrow \) likely
- Long future, immediate loss \( \square / \text{gain} \square \) due to mk-to-mkt
- Invested at higher \( \square / \text{lower} \square \) than average rate
- Similarly when \( S \downarrow \)
  - Positive \( \text{correlation} \ (S, r) \Rightarrow \text{long future} \left\{ \begin{array}{ll}
    \text{more} \square & / \text{less} \square \\
    \text{more} \square & / \text{less} \square
  \end{array} \right. \) attractive than long forward

**Code**

**Output**

![Figure 5.3: Bivariate Gaussian density function. A model for the future, as yet unknown, values of the asset price \( S \) and the short rate \( r \) is for the bivariate probability density function for their log returns to have this form. When the correlation is positive (lhs) we expect futures prices to exceed forward prices. When the correlation is negative (lhs) the reverse is true.](image-url)
Other factors
- taxes, transactions costs, treatment of margins
- credit risk

Equal?
- For short maturities
- Hull $F_0$ is fwd / fut price
- However, Chapter 6, Eurodollar futures 10 yr maturities

Proof that forward and futures prices are equal when interest rates are constant

Proposition

**Proposition 5.10.** A sufficient condition for forward and futures prices to be equal is that interest rates be constant.

Notation

Proof

- Strategy:
  - take a long futures position of $e^{d}$ at the beginning of day 0
  - increase position to $e^{2d}$ at the beginning of day 1
  - ...
  - long futures position $e^{(i+1)d}$ at start of day $i$

- Profit
  - on day 1 (end of day 0) is $(F_1 - F_0) e^d$
  - on day $i$ is $(F_i - F_{i-1}) e^{di}$
  - and is banked

- Compounded value from day $i$ on day $n$ is
  \[(F_i - F_{i-1}) e^{di} e^{δ(n-i)} = (F_i - F_{i-1}) e^{δn}\]

Table 5.7. Dynamic investment strategy in futures contracts
<table>
<thead>
<tr>
<th>Day</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>n−1</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Futures price</td>
<td>$F_0$</td>
<td>$F_1$</td>
<td>$F_2$</td>
<td>…</td>
<td>$F_{n-1}$</td>
<td>$F_n$</td>
</tr>
<tr>
<td>Futures posn</td>
<td>$e^\delta$</td>
<td>$e^{2\delta}$</td>
<td>$e^{3\delta}$</td>
<td>…</td>
<td>$e^{n\delta}$</td>
<td>0</td>
</tr>
</tbody>
</table>

**Value at day $n$ of entire strategy**

$$
\sum_{i=1}^{n} (F_i - F_{i-1}) e^{\delta n} = ((F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1})) e^{\delta n}
$$

$$
= (F_n - F_0) e^{\delta n}
$$

$$
= (S_T - F_0) e^{\delta n}
$$

- **Cost of each increment to the futures position is**
- **Combined strategy of**
  - dynamic strategy above (costs zero, payoff $(S_T - F_0) e^{\delta n}$)
  - invest $F_0$ in a risk-free bank account (costs $F_0$ at 0, pays off $F_0 e^{\delta n}$ at expiry)
- **Total cost at 0 is** $F_0$; **total payoff at** $n$ is $S_T e^{\delta n}$

**Table 5.8. Combined investment strategy: dynamic futures strategy above + bank**

<table>
<thead>
<tr>
<th>Description</th>
<th>Cost (PV at 0)</th>
<th>Payoff (at $n$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic futures strategy</td>
<td>0</td>
<td>$(S_T - F_0) e^{\delta n}$</td>
</tr>
<tr>
<td>Bank account, holding $F_0$</td>
<td>$F_0$</td>
<td>$F_0 e^{\delta n}$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>$F_0$</strong></td>
<td><strong>$S_T e^{\delta n}$</strong></td>
</tr>
</tbody>
</table>

**Table 5.9. Investment strategy: long forward contract + bank**

<table>
<thead>
<tr>
<th>Description</th>
<th>Cost (at 0 or other times)</th>
<th>Payoff (at $n$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long forward 1 unit</td>
<td>0</td>
<td>$(S_T - G_0) e^{\delta n}$</td>
</tr>
<tr>
<td>Bank account, holding $G_0$</td>
<td>$G_0$</td>
<td>$G_0 e^{\delta n}$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>$G_0$</strong></td>
<td><strong>$S_T e^{\delta n}$</strong></td>
</tr>
</tbody>
</table>

- **Both strategies have the same payoff after** $n$ **days, so must be worth the same at time 0**
- $F_0 = G_0$
Futures prices of stock indices

- Can be viewed as an investment asset paying a dividend yield
- Futures / spot price relationship

\[ F_0 = S_0 e^{(r-q)T} \]

where \( q \) is the average dividend yield on the portfolio represented by the index during life of contract

- To be true, index represents an investment asset
- Changes in the index ↔ changes in value of tradable portfolio
- Nikkei index viewed as a dollar number not investment asset (“quanto”)

Index Arbitrage

- \( F_0 > S_0 e^{(r-q)T} \) arbitrageur buys the stocks underlying the index and sells futures
- \( F_0 < S_0 e^{(r-q)T} \) arbitrageur buys futures and shorts or sells the stocks underlying the index
- Involves simultaneous trades in futures and many different stocks
- Often use computer
- Occasionally (e.g., on Black Monday) simultaneous trades are not possible
- Theoretical no-arbitrage relationship between \( F_0 \) and \( S_0 \) fails

Forward and futures prices on currencies

- Foreign currency analogous to security providing dividend yield
- Continuous dividend yield is _____________________

Notation

- \( T \) time until delivery
- \( S_0 \) spot exchange rate, ($ per unit foreign currency)
- \( F_0 \) forward or futures exchange rate, today
- \( r \) domestic ($ ) interest rate
- \( r_f \) foreign interest rate

Proposition

**Proposition 5.11.** The initial forward price \( F_0 \) and spot price \( S_0 \) for a currency for which the foreign interest rate is \( r_f \) are related by \( F_0 = S_0 e^{(r-r_f)T} \)

\[ F_0 = S_0 e^{(r-r_f)T} \] (5.8)
Interest rate parity

- **Interest-rate parity relationship**

<table>
<thead>
<tr>
<th>Time</th>
<th>Foreign FX</th>
<th>Dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>(S_0)</td>
</tr>
<tr>
<td>(T)</td>
<td>(e^{r\cdot T})</td>
<td>(F_0 \cdot e^{r\cdot T} = S_0 \cdot e^{r\cdot T})</td>
</tr>
</tbody>
</table>

Figure 5.4: Two ways of converting a single unit of a foreign currency to dollars at time \(T\).

**Example 5.6.** Two-year interest rates in Australia and the US are 5% and 7%, respectively. The spot FX is 0.62 USD per AUD.

- Find the two-year forward exchange rate.
- Explain a strategy that can be used to establish the interest-rate parity relationship under the assumption of there being no arbitrage opportunities in the market.
- Describe specific strategies to use to exploit forward exchange rates that are
  i) more @ 0.66
  ii) less @ 0.63
  than the theoretical forward price that you have already calculated.

\[
0.62 \cdot e^{0.07 - 0.05 \cdot 2} = 0.6453
\]

<table>
<thead>
<tr>
<th>Instrument</th>
<th>#</th>
<th>Value at 0 ($)</th>
<th>Value at (T) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foreign bond</td>
<td>1</td>
<td>(S_0)</td>
<td>(S_T \cdot e^{r\cdot T})</td>
</tr>
<tr>
<td>Domestic bond</td>
<td>(-S_0)</td>
<td>(-S_0)</td>
<td>(-S_0 \cdot e^{r\cdot T})</td>
</tr>
<tr>
<td>Forward FX</td>
<td>(-e^{r\cdot T})</td>
<td>0</td>
<td>(-e^{r\cdot T} (S_T - F_0))</td>
</tr>
<tr>
<td>Total</td>
<td>0</td>
<td>0</td>
<td>(F_0 \cdot e^{r\cdot T} - S_0 \cdot e^{r\cdot T})</td>
</tr>
</tbody>
</table>

i) As in the table above, go long and short the foreign and domestic bonds in the ratio 1: \(S_0\).

We are long the foreign FX, so short the fwd, i.e. sell AUD in the future

**Foreign bond investment** grows to \(e^{r\cdot T}\) in the foreign currency

**Domestic bond debt** grows to \(S_0 \cdot e^{r\cdot T}\) in the domestic currency

Convert foreign investment to domestic FX using forward (sell foreign), raising \(F_0 \cdot e^{r\cdot T}\)

Pay off domestic debt of \(S_0 \cdot e^{r\cdot T}\)

Profit is

\[
L \left( F_0 \cdot e^{r\cdot T} - S_0 \cdot e^{r\cdot T} \right) = \frac{1612 \text{ AUD}}{0.62} \times (0.66 \cdot e^{0.05 \cdot 2} - 0.62 \cdot e^{0.07 \cdot 2}) = \$16.91
\]
\[
\frac{1000}{0.62} (0.66 e^{0.05 \times 2} - 0.62 e^{0.07 \times 2})
\]

26.1985

ii) Reverse signs in the table above, go short and long the foreign and domestic bonds in the ratio 1: S0.

... Profit is \(-1000 (0.63 e^{0.05 \times 2} - 0.62 e^{0.07 \times 2})\)

\[-1000 (0.63 e^{0.05 \times 2} - 0.62 e^{0.07 \times 2})
\]

16.9121

As ever, note the opposite sign between the asset (in this case foreign FX) underlying the future and the future

---

**Futures on commodities**

**Income and storage costs**

- Gold and silver are _______ assets, which means _______
- *Gold lease rate* is interest earned for lending gold
- Gold has storage costs too

**Notation**

\[
U \quad \text{present value of storage costs over life of forward contract}
\]

\[
u \quad \text{storage costs proportional to cost of commodity, to be treated as negative yield}
\]

**Known PV of storage costs**

- Treat storage costs as negative income

**Proposition 5.12.** The initial forward price \(F_0\) and spot price \(S_0\) for a consumption asset for which the present value of the storage costs are \(U\) satisfy \(F_0 = (S_0 + U) e^{rT}\)

\[
F_0 = (S_0 + U) e^{rT}
\]

(5.9)

**Storage costs proportional to commodity price**

- Treat storage costs as negative yield
Proposition 5.13. The initial forward price $F_0$ and spot price $S_0$ for a consumption asset for which the storage costs per unit time is $u$ satisfy $F_0 = S_0 e^{(r+u)T}$

\[ F_0 = S_0 e^{(r+u)T} \] (5.10)

Consumption commodities

Known PV of storage costs

- Treat storage costs as negative income

Proposition 5.14. The initial forward price $F_0$ and spot price $S_0$ for a consumption asset for which the present value of the storage costs are $U$ obey the inequality $F_0 \leq (S_0 + U) e^{rT}$

\[ F_0 \leq (S_0 + U) e^{rT} \] (5.11)

Storage costs proportional to commodity price

- Treat storage costs as negative yield

Proposition 5.15. The initial forward price $F_0$ and spot price $S_0$ for a consumption asset for which the storage costs per unit time is $u$ obey the inequality $F_0 \leq S_0 e^{(r+u)T}$

\[ F_0 \leq S_0 e^{(r+u)T} \] (5.12)

Proof: see example

Example 5.7. Describe a strategy to exploit the arbitrage opportunity that exists when the parameters for a commodity with forward price $F_0$ for maturity $T$, spot price $S_0$, storage costs of present value $U$, satisfy $F_0 > (S_0 + U) e^{rT}$. What will be the effect of the actions in the market place by arbitrageurs? If the commodity is a consumption asset, can investors also profit risklessly when $F_0 < (S_0 + U) e^{rT}$? If you think not, explain why the strategy that you would use for investment assets for futures prices relatively high with respect to spot prices is not effective.

Refer back to strategies for investment strategies. (here)

No. Companies with inventory reluctant to sell commodity & buy fwd. Fwds cannot be consumed!
Inequality is the strongest relationship that we can deduce by no arb args.

Convenience yields

- Benefits of commodity cf. forward
  - keep production running
  - profit from shortages
- E.g. ______ is a consumption asset
- Reflect market’s view on the future __________ of the commodity
- The greater the possibility of ___________, the __________ the CY
- Inventories of users

**Notation**

| y  | convenience yield |

**Definition 5.16.** The *convenience yield* is the value of $y$ such that when the storage costs are known and have present value $U$, then $F_0 e^{yT} = (S_0 + U) e^{rT}$. Similarly for storage costs that are a constant proportion $u$ of the spot price: $F_0 e^{yT} = S_0 e^{(r+u)T}$.

$$F_0 e^{yT} = (S_0 + U) e^{rT}$$
$$F_0 = S_0 e^{(r+u)T}$$

(5.13)

Convenience yield measures extent to which forward price of consumptions assets falls short of the theoretical value for investment assets

- The convenience yield for investment assets is ______

**Figure**

**Code**

**Output**

![Graph showing futures price as a function of time to maturity for gold (lhs) and oil (rhs)](image.png)

**Figure 5.5:** Futures price as a function of time to maturity for gold (lhs) and oil (rhs)

**Example 5.8.** What can we deduce from the diagram about the relative size of the convenience yield $y$ and the sum of the interest rate and storage cost rate $r + u$?

Deduce $y$ is greater ☐ less ☐ than $r + u$
The cost of carry

Notation

\[ c \quad \text{cost of carry} \]

Definition

**Definition 5.17.** The *cost of carry* is the storage cost plus the interest costs less the income earned.

\[ c = r + u - q \quad (5.14) \]

**Table 5.10.** Cost of carry for various assets

<table>
<thead>
<tr>
<th>Asset</th>
<th>Cost of carry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-div paying stock</td>
<td>( r )</td>
</tr>
<tr>
<td>Stock index</td>
<td>( r-q )</td>
</tr>
<tr>
<td>Currency</td>
<td>( r-r_f )</td>
</tr>
<tr>
<td>Commodity</td>
<td>( r-q+u )</td>
</tr>
</tbody>
</table>

Relationships between forward and spot prices in terms of the cost of carry

**Investment asset**

**Proposition 5.18.** The initial forward price \( F_0 \) and spot price \( S_0 \) for an investment asset that pays no dividend are related by \( F_0 = S_0 e^{cT} \), where...

\[ F_0 = S_0 e^{cT} \quad (5.15) \]

**Consumption asset**

**Proposition 5.19.** The initial forward price \( F_0 \) and spot price \( S_0 \) for a consumption asset that pays no dividend are related by \( F_0 = S_0 e^{(c-y)T} \), where...

\[ F_0 = S_0 e^{(c-y)T} \quad (5.16) \]
Delivery options

- Party with ________ position gets to choose when to deliver
- When \( \frac{c^c}{y^y} > \frac{c^c}{y^y} \), forward curve is an \( _______ \) function of maturity, and it is best to deliver \( _______ \). Why?

Futures prices and expected future spot prices

Notation

\[ k \] expected return required by investors on an asset

Strategy

- Invest
  - \( F_0 e^{-rT} \) at the risk-free rate
  - long futures contract \( \rightarrow \) cash inflow of \( S_T \) at maturity
- Systematic risk ( ________ with ________ ) of asset:
  - none: \( k = r \), \( F_0 \) is an unbiased estimate of \( S_T \)
  - positive: \( k > r \), \( F_0 < \mathbb{E}(S_T) \)
  - negative: \( k < r \), \( F_0 > \mathbb{E}(S_T) \)

Normal backwardation and contango

- \( F_0 < \mathbb{E}(S_T) \) normal backwardation
- \( F_0 > \mathbb{E}(S_T) \) contango

Summary

- Forward and futures prices same? Nearly
- Exactly when IRs deterministic
- Investment vs consumption assets
- Investment assets, cases. Asset provides
  - None
  - Known $  
  - Known yield
Table 5.11. Summary table of formulae used to find the forward price and the value of a forward contract, for the three cases in which there is no income, a known income with present value $I$, and a known yield $y$.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Forward / futures price</th>
<th>Value of long forward contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>No income</td>
<td>$S_0 : e^{rT}$</td>
<td>$S_0 – K : e^{-rT}$</td>
</tr>
<tr>
<td>Income of present value $I$</td>
<td>$(S_0 – I) : e^{rT}$</td>
<td>$S_0-I-K : e^{-rT}$</td>
</tr>
<tr>
<td>Yield $q$</td>
<td>$S_0 : e^{(r-q)T}$</td>
<td>$S_0 : e^{-qT}-K : e^{-rT}$</td>
</tr>
</tbody>
</table>

- Find futures prices for
  - stock indices
  - currencies
  - gold and silver
- Consumption assets – futures not a function of spot + observable vars
- Can get upper bound
- Convenience yield – owning commodity better than owning future
- Benefits
  - profit from temp shortages
  - keep production process running
- Cost of carry
  - + storage costs
  - + financing
  - – income
- Futures price > spot price
  - Investment – cost of carry
  - Consumption – cost of carry, net convenience yield